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Introduction to wave equations for lossy media

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Wave equation

- This is *the* equation in array signal processing.
- Lossless wave equation

$$\nabla^2 \vec{s} = \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 s}{\partial t^2}$$

- $\Delta = \nabla^2$ is the Laplacian operator (del=nabla squared)
- *s* = s(x,y,z,t) is a general scalar field (electromagnetics: electric or magnetic field, acoustics: sound pressure ...)
- *c* is the speed of propagation

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Three standard coupled acoustic equations

1. The linearized equation of motion (Euler's inviscid force equation, Newton's 2. law F=ma)
$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\nabla p$$
2. The linearized equation of continuity (conservation of mass)
$$\frac{\partial \rho}{\partial t} = -\rho_0 \nabla \cdot u$$
3. The adiabatic equation of state (Hooke's law *F=kx)*
$$p = c_0^2 \rho$$

- Here ***u*** is the acoustic velocity, and *ρ* and ρ0 are the acoustic and ambient density, respectively.

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Three simple principles behind the acoustic wave equation

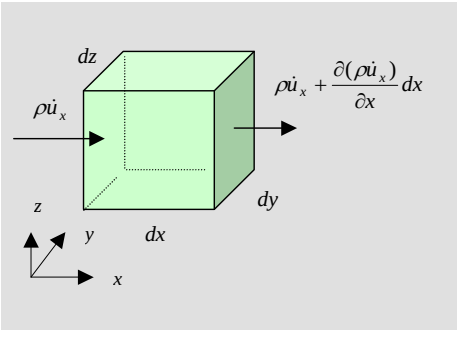

Figure: J Hovem, TTT4175 Marin akustikk, NTNU

1. Equation of continuity: conservation of mass
2. Newton's 2. law: $F = m\,a$
3. State equation: relationship between change in pressure and volume (in one dimension this is Hooke's law: $F = k\,x$ – spring)

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Solution

Guesses:

1. Separable $s(x,y,z,t) = A \cdot s_t(t) \cdot s_x(x) \cdot s_y(y) \cdot s_z(z)$
2. Complex exponential in time: $s(t) = \exp\{j\omega t\}$
3. Complex exponential in space
   $s_x(x) = \exp\{-jk_x t\}$, (also in y and z)

Assumed solution:

$$s(\vec{x}, t) = A \exp\{j(\omega t - k_x \cdot x - k_y \cdot y - k_z \cdot z)\}$$

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Solution

Insert $s(\vec{x}, t) = A \exp\{j(\omega t - k_x \cdot x - k_y \cdot y - k_z \cdot z)\}$

into $\nabla^2 \vec{s} = \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 s}{\partial t^2}$

$\Rightarrow$ $k_x^2\, s(\cdot) + k_y^2 s(\cdot) + k_z^2\, s(\cdot) = \omega^2 s(\cdot)/c^2$

or $k_x^2 + k_y^2 + k_z^2 = |k|^2 = \omega^2/c^2$ or $|k| = \omega/c$

which is the condition for this guess to be a solution

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Deviations from simple media

1. Dispersion: $c = c(\omega)$
   - Group and phase velocity, dispersion equation: $\omega = f(k) \neq c \cdot k$
   - Evanescent (= non-propagating) waves: purely imaginary k
2. Loss: $c = c_{\Re} + jc_{\Im}$
   - Wavenumber is no longer real, imaginary part gives attenuation.
   - Waveform changes with distance
3. Non-linearity: c = c(s(t))
   - Generation of harmonics, shock waves
4. Refraction, non-homogenoeus medium: c=c(x,y,z)
   - Snell's law

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Requirements of a wave equation

- Fit the real world and measurements
- Satisfy basic properties like causality
- Based on constitutive equations
- Simplicity and beauty

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"It is more important to have beauty in one's equations than to have them fit experiment

... because the discrepancy may be due to minor features which are not properly taken into account and which will get cleared up with further developments of the theory ...

It seems that if one is working from the point of view of getting beauty in one's equations, and if one has a really sound instinct, one is on a sure line of success."

Dirac, "The Evolution of the Physicist's Picture of Nature," *Scientific American*, May 1963.

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Euler's identity

$$e^{i\pi} + 1 = 0$$

- "The most remarkable formula in mathematics"
  - According to Richard Feynman
- "the Most Beautiful Mathematical Formula Ever"
  - Voted by readers of the Mathematical Intelligencer in 1988

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God as a mathematician

- Eugene Wigner, The Unreasonable Effectiveness of Mathematics in the Natural Sciences, Comm. Pure Appl. Math, 1960
  - http://www.dartmouth.edu/~matc/MathDrama/reading/Wigner.html
- Why does the ratio of the circumference of a circle and its diameter have anything to do with Gaussian noise?

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

π

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Viscous wave equation

Additional loss term

- Sound in a viscous fluid, augmented wave eq.:

$$\nabla^2 \vec{s} = \frac{1}{c^2}\frac{\partial^2 \vec{s}}{\partial t^2} - \frac{4\mu}{3\rho_0 c^2}\frac{\partial}{\partial t}\nabla^2 \vec{s} = \frac{1}{c^2}\frac{\partial^2 \vec{s}}{\partial t^2} - \tau\frac{\partial}{\partial t}\nabla^2 \vec{s}$$

  - μ is shear bulk viscocity coefficient
  - τ is a relaxation time
  - Johnson & Dudgeon, problem 2.7
- Approximate solution (low frequency, low loss):

$$k_{\Im} \approx -\frac{\tau}{2c}\omega^2$$

- Attenuation that increases with ω^2

Dispersion relation

- Viscoelastic wave equation: $\nabla^2 u - \frac{1}{c_0^2}\frac{\partial^2 u}{\partial t^2} + \tau\frac{\partial}{\partial t}(\nabla^2 u) = 0$
- Assume 1-D, and u(x,t)=exp(j(ωt-kx)):

$$(-jk)^2 u(x,t) - \frac{(j\omega)^2}{c_0^2}u(x,t) + \tau(j\omega(-jk)^2)u(x,t) = 0$$

$$k^2 - \frac{\omega^2}{c_0^2} + j\omega\tau k^2 = 0$$

- $k = k_{\Re} + jk_{\Im} = \beta - j\alpha \Rightarrow u = \exp(-\alpha x)\cdot\exp(j(\omega t - \beta x))$
- Let $\omega\tau \ll 1$ and solve for k: $\alpha \approx \frac{\tau}{2c_0}\omega^2$

See solution of problem 2.7 for more details